SCHLÄFLI MODULAR EQUATIONS FOR GENERALIZED WEBER FUNCTIONS

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ABSTRACT. Sets of appropriately normalized eta quotients, that we call level n Weber functions, are defined, and certain identities generalizing Weber function identities are proved for these functions. Schläfli type modular equations are explicitly obtained for Generalized Weber Functions associated with a Fricke group $\Gamma^0(n)^+$, for $n = 2, 3, 5, 7, 11, 13$ and 17.

INTRODUCTION

The main purpose of this paper is to describe the explicit construction of modular equations for the functions

$$\mathfrak{m}_{n,0}(\tau) = \frac{\eta(\tau/n)}{\eta(\tau)}, \tag{1}$$

for various $n \in \mathbb{N}$, where $\eta(\tau)$ is the well known Dedekind eta function. These functions $\mathfrak{m}_{n,0}(\tau)$ have been called *Generalized Weber Functions* and have been investigated in various contexts before (see for example [11] or even [17]).

We are not the first to consider Schläfli type equations since Weber. Modular equations of this type have also been found by Watson [16]. Modular equations also remain a topic of active interest; see for example the work of Chan and Liaw [5], [6]. For more information on the literature associated with modular equations and class invariants, the reader can very profitably consult Berndt's book [2].

In some ways our treatment of this topic will resemble that of Weber [17] who obtained Schläfli type modular equations for his *Weber functions*, of which

$$\mathfrak{f}_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)},$$

is our function $\mathfrak{m}_{n,0}(\tau)$ for $n = 2$.

Despite some similarity however, there will be a number of interesting differences in both the results and the methods employed. For instance, we will obtain modular equations of both prime and composite *degree*[1] for generalized Weber functions, with only the restriction that the degree m be coprime to n.

Also, Weber's approach to modular equations is via the theory of theta functions, which is still of some interest, however we will make use of the theory of modular functions, in particular functions for the congruence subgroups $\Gamma^0(n)$.

Of course various other arithmetic oddities which do not appear in the case dealt with by Weber ($n = 2$) introduce additional interesting twists along the way. For example, in most cases Schläfli type modular equations are equivalent to the minimal, irreducible polynomial relationship $P(u, v) = 0$ between u (our function $\mathfrak{m}_{n,0}$

[1]A modular equation is generally a polynomial relationship, $P(u, v) = 0$, between two functions $u(\tau)$ and $v = u(m\tau)$, for some *degree* $m \in \mathbb{N}$, not to be confused with the degree of the polynomial $P(u, v)$ itself.

defined above) and $v = u(m\tau)$. However in some exceptional cases such minimal modular equations are not Schläfli. This reflects in the theory at precisely the points where additional "tricks" have to be employed to obtain a Schläfli modular equation.

Our generalization of Weber's work will not however be quite all encompassing. Apart from the condition already mentioned that $(m, n) = 1$, we will not deal with the cases where the level n is composite, since in those cases $\Gamma^0(n)$ has more than two inequivalent cusps, thereby multiplying the effort required in rigorously establishing modular equations. In fact in many cases the method oulined in this paper yields modular equations for composite level anyway, and one can then check the extra cusps by hand. But for reasons of time and space we make the restriction that n will be prime.

Before coming to the details of the main method employed in this paper, we give the following explicit definition of Schläfli modular equations. Fix a level n, and denote our basic function (1), for now, by $u(\tau)$. For a chosen *degree* $m \in \mathbb{N}$ we define functions

$$P(\tau) = (u(\tau)u(m\tau))^k \text{ and } Q(\tau) = \left(\frac{u(\tau)}{u(m\tau)}\right)^l$$

for a certain pair of natural numbers k and l, dependent on the degree m.

A Schläfli modular equation in this context is then a polynomial relation between two functions of the form

$$A = P + c/P \text{ and } B = Q \pm 1/Q,$$

for some $c \in \mathbb{R}$ and where the sign in B also depends on the degree m.

As mentioned, in this paper we obtain explicit Schläfli modular equations for various prime levels n and for many degrees m, both prime and composite.

The method of obtaining such modular equations is essentially by constructing modular functions for the Fricke group $\Gamma^0(n)^+$. This is defined to be the subgroup of $SL_2(\mathbb{R})$ generated by the congruence subgroup

$$\Gamma^0(n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \beta \equiv 0 \pmod{n} \right\},$$

along with the Fricke involution $\frac{1}{\sqrt{n}}\begin{pmatrix} 0 & -n \\ 1 & 0 \end{pmatrix}$ added in. Note that the complex upper half plane modulo the action of the Fricke group can be compactified and made into a Riemann surface and the index of the group $\Gamma^0(n)$ in the Fricke group $\Gamma^0(n)^+$ is two.

Modular functions for such $\Gamma^0(n)^+$ can then be thought of as meromorphic functions on this associated Riemann surface.

To create a modular equation for $\mathfrak{m}_{n,0}$ of degree m we construct a family of related functions F_c which satisfy the conditions below. We list this set of conditions firstly for the case where the degree m is prime and follow this by describing appropriate modifications for the composite case. Recall that the level n will be taken to be prime when constructing modular equations. The recipe for creating modular equations under such hyotheses is:

(i) The F_c are modular functions (it is not necessary to specify a level at this point);

(ii) They are permuted by $\Gamma^0(n)$ (thus symmetric combinations of them have level n);

(iii) They have no poles in the complex upper half plane;

(iv) They are permuted at least up to sign by the Fricke involution, which takes the cusp $\tau = 0$ to the 'other' (inequivalent) cusp of $\Gamma^0(n)$ for $n > 1$, $\tau = i\infty$;

(v) The q-series of one of the functions, F_∞, vanishes up to and including the constant term;

(vi) The vanishing of the q-series of one of the F_c, as per (v), implies the vanishing of the q-series of the other F_c, so that the product $G = \prod_c F_c$ has its q-series vanish up to and including the constant term; and

(vii) The functions F_c (including F_∞) are related, by certain transformations of the complex upper half plane, in such a way that the vanishing of one identically (as a function of τ), implies the vanishing of all the others identically.

Note that (i) and (ii) imply that $G = \prod_c F_c$ is a modular function for $\Gamma^0(n)$. Then (iv), (v) and (vi) together imply that G^2, and therefore also G, has a zero at $\tau = 0$ and $\tau = i\infty$.

Combining these facts with (iii) and Liouville's Theorem we see that G is a constant, which in this case is clearly zero. Thus one of the factors of G is zero. This, with (vii), implies that all the F_c are identically zero.

We will achieve all of the above in practice by specifying sets of functions A_c and B_c which are permuted in the same way by $\Gamma^0(n)$ and up to sign by the Fricke involution. We will then define $F_c = F(A_c, B_c)$ for some carefully chosen polynomial $F(x, y) \in \mathbb{Z}[x, y]$.

In fact A_∞ and B_∞ will be the functions A and B mentioned earlier involving only $\mathfrak{m}_{n,0}(\tau)$ and $\mathfrak{m}_{n,0}(m\tau)$. As we just showed, (i) to (vii) then guarantee that F_∞ is identically zero. But this provides us with a Schläfli modular equation, for $F_\infty = 0$ induces (from the definition of A and B) a polynomial relationship between A_∞ and B_∞.

For the case where the degree m is composite, this recipe breaks down. In fact we find that there are additional functions F_c that appear in the theory, associated to each of the non-trivial factors of m. It turns out that these new functions F_c are not related to the original functions F_c, which were associated to the trivial factors of m. This in turn makes it impossible to guarantee the condition (vi) above.

This problem is rectified by computing the leading powers of the q-series of the additional functions F_c and determining a bound on their combined contribution to the q-series of $G = \prod_c F_c$. One then arranges for the q-series of G to vanish as before, by increasing the strength of condition (v) above. One requires that the q-series of F_∞ vanishes sufficiently far to ensure that the contribution of all the additional F_c has been compensated for.

This is sufficient to guarantee a modular equation $F_\infty = 0$ as before, provided that none of the new functions F_c vanish identically. This is easy to check in practice by computing q-series or by substituting a random value of τ and seeing that the new F_c don't vanish. Because of relationships amongst the new F_c, implicit in the sequel, it is only necessary to test a very small number of them in this way, therefore nothing more will be said about this in what follows.

The author's interest in these modular equations arose from a study of the use of modular equations in evaluating singular values of quotients of the Dedekind eta function. These turn out to provide explicit generators for ring class fields of certain imaginary quadratic number fields (see [12]).

The final section of this paper details a simple evaluation along these lines by making use of the modular equations derived earlier. Further details and more interesting examples of this technique can be found in the author's thesis [13] and in [9].

1. Generalized Weber Functions

We begin by demonstrating that the functions (1) really have the right to be called Generalized Weber functions. We do this by not just exhibiting a single function for each level n as we have above but by defining a whole set of functions for each level, which we will call the 'level n Weber functions'. Then we show that each such set of functions has properties very similar to the set of classical Weber functions (which can be thought of as level two Weber functions).

For simplicity we will restrict to the cases where the level n is prime.

It is well known that for a prime n the set of functions

$$f_\infty(\tau) = n^{12}\frac{\Delta(n\tau)}{\Delta(\tau)} \quad \text{and} \quad f_j(\tau) = \frac{\Delta((\tau+j)/n)}{\Delta(\tau)}; \quad 0 \le j \le n-1, \tag{2}$$

is merely permuted by any transformation of the full modular group. Thus this collection of functions is 'complete' in the sense that no transform of such a function by a modular transformation is new.

However our context is that of modular equations, and Δ-quotients are too 'large' for our purposes, yielding cumbersome coefficients. The Dedekind eta function $\eta(\tau)$ is a 24-th root of the Δ-function and so it makes sense to consider 24-th roots of the functions above. But then we have a choice of normalization by 24-th roots of unity.

We take the following normalization

$$\begin{aligned} \mathfrak{m}_{n,\infty}(\tau) &= \sqrt{n}\,\frac{\eta(n\tau)}{\eta(\tau)}, \quad \mathfrak{m}_{n,0}(\tau) = \frac{\eta\left(\frac{\tau}{n}\right)}{\eta(\tau)}, \\ \mathfrak{m}_{n,j}(\tau) &= \zeta_{24}^{n-j-1}\,\frac{\eta\left(\frac{\tau+j}{n}\right)}{\eta(\tau)}; \quad 1 \le j \le n-1, \end{aligned} \tag{3}$$

where $\zeta_n = \exp(2\pi i/n)$. We will refer to these as the *level n Weber functions* or occasionally as the *level n functions*. As we shall see later, this is a slight abuse of language since it is actually certain powers of these functions which are modular functions of level n.

Firstly we will derive the modular transformation laws and various other identities for (3). Before doing this in general however we give the reader a chance to become more acquainted with the various sets of functions we have defined, and to avoid a plethora of subscripts, by explicitly setting out the definition of the level 3 functions and their various identities, and similarly for the level 5 functions. Note that the proofs of the actual identities for these cases will not be given immediately but will be deferred until the situation for a general prime level n is investigated.

1.1. Level Three Functions.

Here we take the 'level' n to be 3 in (3). We denote the functions $\mathfrak{m}_{3,i}$ by $\mathfrak{g}_i$.

Definition 1.1.1. *The four level 3 functions are defined to be*

$$\mathfrak{g}_\infty(\tau) = \sqrt{3}\,\frac{\eta(3\tau)}{\eta(\tau)}, \quad \mathfrak{g}_0(\tau) = \frac{\eta(\frac{\tau}{3})}{\eta(\tau)}, \quad \mathfrak{g}_1(\tau) = \zeta_{24}\,\frac{\eta(\frac{\tau+1}{3})}{\eta(\tau)}, \quad \mathfrak{g}_2(\tau) = \frac{\eta(\frac{\tau+2}{3})}{\eta(\tau)}.$$

We should note immediately that various functions of this type already appear in the literature (see for example [12] or even [17] §72), however they have different normalizations to ours (or none at all).

Essentially one can choose either to normalize so that analogues of various ordinary Weber function identities are as 'elegant' as possible, or so that the modular transformation laws are as simple as possible, but not both. Here we have chosen to exhibit the latter, for comparison with what appears already in the literature.

The modular transformation laws can be written as follows.

Theorem 1.1.2.

$$\begin{pmatrix} \mathfrak{g}_\infty \\ \mathfrak{g}_0 \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \end{pmatrix} \circ T = \begin{pmatrix} \zeta_{12}\,\mathfrak{g}_\infty \\ \zeta_{12}^{-1}\,\mathfrak{g}_1 \\ \mathfrak{g}_2 \\ \mathfrak{g}_0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \mathfrak{g}_\infty \\ \mathfrak{g}_0 \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \end{pmatrix} \circ S = \begin{pmatrix} \mathfrak{g}_0 \\ \mathfrak{g}_\infty \\ \mathfrak{g}_2 \\ \mathfrak{g}_1 \end{pmatrix}$$

where T stands for the transformation $\tau \to \tau + 1$ and S for $\tau \to -1/\tau$.

Note: It will also be convenient to let T and S denote matrices associated to these fractional linear transformations. To this end all matrices and congruence subgroups in this paper will be thought of as belonging to the inhomogeneous modular group $\Gamma = SL_2(\mathbb{Z})/\{\pm I\}$ where I is the 2×2 identity matrix. (For more details on this see [15] I §2.)

We also have the following identities

Theorem 1.1.3. *The product of the four functions $\mathfrak{g}_i$ is a constant on the complex upper half plane:*

$$\mathfrak{g}_\infty(\tau)\,\mathfrak{g}_0(\tau)\,\mathfrak{g}_1(\tau)\,\mathfrak{g}_2(\tau) = \zeta_{12}\,\sqrt{3}.$$

Theorem 1.1.4. *We have*

$$\mathfrak{g}_\infty(\tau)^6 - \mathfrak{g}_0(\tau)^6 - \mathfrak{g}_1(\tau)^6 + \mathfrak{g}_2(\tau)^6 = 0.$$

1.2. **Level Five Functions.** Here we take $n = 5$ in (3).

Definition 1.2.1. *The six level 5 Weber functions are defined to be*

$$\mathfrak{h}_\infty(\tau) = \sqrt{5}\,\frac{\eta(5\tau)}{\eta(\tau)}, \quad \mathfrak{h}_0(\tau) = \frac{\eta(\frac{\tau}{5})}{\eta(\tau)}, \quad \mathfrak{h}_1(\tau) = \zeta_{24}^3\,\frac{\eta(\frac{\tau+1}{5})}{\eta(\tau)},$$

$$\mathfrak{h}_2(\tau) = \zeta_{24}^2\,\frac{\eta(\frac{\tau+2}{5})}{\eta(\tau)}, \quad \mathfrak{h}_3(\tau) = \zeta_{24}\,\frac{\eta(\frac{\tau+3}{5})}{\eta(\tau)}, \quad \mathfrak{h}_4(\tau) = \frac{\eta(\frac{\tau+4}{5})}{\eta(\tau)}.$$

For these functions we have the following.

Theorem 1.2.2.

$$\begin{pmatrix} \mathfrak{h}_\infty \\ \mathfrak{h}_0 \\ \mathfrak{h}_1 \\ \mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \end{pmatrix} \circ T = \begin{pmatrix} \zeta_6\,\mathfrak{h}_\infty \\ \zeta_6^{-1}\,\mathfrak{h}_1 \\ \mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \\ \mathfrak{h}_0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \mathfrak{h}_\infty \\ \mathfrak{h}_0 \\ \mathfrak{h}_1 \\ \mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \end{pmatrix} \circ S = \begin{pmatrix} \mathfrak{h}_0 \\ \mathfrak{h}_\infty \\ \mathfrak{h}_4 \\ \mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_1 \end{pmatrix}.$$

Note that here and in Theorem 1.1.2 only two roots of unity appear in each case amongst the modular transformation rules. This is a feature of our normalization for all of the levels $n = 2, 3, 5, 7$ and 13, i.e. those prime level n cases where $\Gamma^0(n)$ is genus zero.

As for level 3 there are some additional identities:

Theorem 1.2.3. *The product of the six functions $\mathfrak{h}_i$ is a constant on the complex upper half plane:*

$$\mathfrak{h}_\infty(\tau)\,\mathfrak{h}_0(\tau)\,\mathfrak{h}_1(\tau)\,\mathfrak{h}_2(\tau)\,\mathfrak{h}_3(\tau)\,\mathfrak{h}_4(\tau) = \zeta_3\,\sqrt{5}.$$

Theorem 1.2.4. *We have*

$$\mathfrak{h}_\infty(\tau)^6 + \mathfrak{h}_0(\tau)^6 + \mathfrak{h}_1(\tau)^6 + \mathfrak{h}_2(\tau)^6 + \mathfrak{h}_3(\tau)^6 + \mathfrak{h}_4(\tau)^6 = -30.$$

Note that this identity differs from the corresponding one for level 3, in that the constant is not zero. We discuss this feature below when we come to prove these identities.

1.3. **Prime Level Weber Functions in General.** We now consider the functions (3) for a general prime level n. In fact we will also let n be odd, since the identities for the level 2 Weber functions can easily be derived from those for the ordinary Weber functions.

The theorem which follows can be proved easily from the first of the transformation laws of the Dedekind eta function, both of which we state here for convenience.

$$\eta(\tau+1) = \zeta_{24}\,\eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \tag{4}$$

where the square root is always taken to have argument between $-\pi/2$ and $\pi/2$.

Theorem 1.3.1. *We have*

$$\mathfrak{m}_{n,i}(\tau+1) = \mathfrak{m}_{n,i+1}(\tau) \ \textit{for}\ 1 \le i \le n-2,$$
$$\mathfrak{m}_{n,n-1}(\tau+1) = \mathfrak{m}_{n,0}(\tau), \quad \mathfrak{m}_{n,0}(\tau+1) = \zeta_{24}^{1-n}\,\mathfrak{m}_{n,1}(\tau)$$
$$\textit{and}\ \ \mathfrak{m}_{n,\infty}(\tau+1) = \zeta_{24}^{n-1}\,\mathfrak{m}_{n,\infty}(\tau).$$

□

Note that if we define

$$s(n) = \frac{24}{\gcd(24, n-1)}, \tag{5}$$

then the roots of unity which appear in this theorem are $s(n)$-th roots of unity. Since n is an odd prime, $s(n) \mid 12$.

The following theorem also follows trivially from the other of the transformation laws of the eta function. We leave the routine verification of this and the previous theorem to the reader.

Theorem 1.3.2. *We have*

$$\mathfrak{m}_{n,\infty}(-1/\tau) = \mathfrak{m}_{n,0}(\tau).$$

□

It now remains to investigate the action of the transformation $S : \tau \to -1/\tau$ on the remaining functions $\mathfrak{m}_{n,c}$ where $c \neq 0, \infty$.

It is valid to ask the question: for which natural numbers n (not necessarily prime) does the transformation S introduce no root of unity factors in the transformation laws of the functions $\mathfrak{m}_{n,c}$; $c \neq \infty$ with $\gcd(n, c) = 1$, given that we extend our definition for these functions to the case of composite n in the obvious way. A computer search reveals that the sequence of such n begins $2-7$, 9, 10, 13 and then probably just all squares of odd primes from there on.

In fact this can be proved by making use of an identity (8) which we are about to prove, if we extend it by allowing composite and even n and demanding c be coprime with n. But it would be a diversion to attempt to prove the full result here.

However for odd primes n, we are left with only $n = 3$, 5, 7 and 13 from this list. We will verify that the result holds for these values.

The main tool which we make use of is a transformation formula of Weber for a general linear transformation of the eta function. In §38 of [17] Weber gives the following formula for such a transformation with associated matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Lemma 1.3.3. *Define a function E by*

$$\eta(A\tau) = E(A;\tau)\,\eta(\tau)$$

then

$$E(A;\tau) = \begin{cases} \left(\frac{\gamma}{\delta}\right) i^{\frac{\delta-1}{2}} \zeta_{24}^{\delta(\beta-\gamma)-(\delta^2-1)\alpha\gamma} \sqrt{\gamma\tau+\delta}, & \text{if } \delta > 0 \text{ is odd} \\ \left(\frac{\delta}{\gamma}\right) i^{\frac{1-\gamma}{2}} \zeta_{24}^{\gamma(\alpha+\delta)-(\gamma^2-1)\beta\delta} \sqrt{-i(\gamma\tau+\delta)}, & \text{if } \gamma > 0 \text{ is odd} \end{cases},$$

involving Jacobi symbols.

Now consider the effect of applying the transformation $S : \tau \to -1/\tau$ to $\mathfrak{m}_{n,c}$ for $c \neq 0, \infty$. The most difficult part is the action of S on $\eta\left(\frac{\tau+c}{n}\right)$. This we can determine by expressing the new argument of the eta function, which results when S is applied to τ, as the composition of a linear transformation and an argument of the form $\frac{\tau+k}{n}$ for some $1 \le k \le n-1$. This is equivalent to solving the following matrix equation.

$$\begin{pmatrix} 1 & c \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix}, \tag{6}$$

for a linear transformation A as above.

We obtain five equations to solve simultaneously:

$$\alpha\delta - \beta\gamma = 1, \quad \alpha = c, \quad -1 = \alpha k + \beta n, \quad n = \gamma \ \text{ and } \ 0 = \gamma k + \delta n.$$

It is easy to see that a solution to these is given by

$$\alpha = c, \quad \beta = -(ck+1)/n, \quad \gamma = n, \quad \delta = -k.$$

Since we require a transformation in $\mathrm{SL}_2(\mathbb{Z})$ we must have that

$$ck \equiv -1 \pmod{n},$$

and since n is a prime and $n \nmid c$, this can be written

$$k \equiv -1/c \pmod{n}.$$

We will of course always pick k to be in the range $1 \le k \le n-1$ so that it can be the subscript of one of our functions $\mathfrak{m}_{n,k}$.

Now we note that $\gamma = n$ is odd and positive and so Weber's formula can be used to determine the action of the linear transformation A on the eta function. The value is

$$E(A;\tau) = \left(\frac{-k}{n}\right) i^{\frac{1-n}{2}} \zeta_{24}^{n(c-k)-(n^2-1)k(ck+1)/n} \sqrt{-i\left(n\left(\frac{\tau+k}{n}\right)-k\right)} \tag{7}$$

$$= \left(\frac{-k}{n}\right) i^{\frac{1-n}{2}} \zeta_{24}^{n(c-k)} \sqrt{-i\tau}, \tag{8}$$

where the last equality holds for $n > 3$ an odd prime, since $24 | n^2-1$ for such primes (it is an easy matter to deal with $n = 3$ separately, since one only needs to look at $c = 1$, $k = 2$).

Now it is an easy matter to verify by hand that for $n = 3, 5, 7$ and 13, we have

$$E(A;\tau) = \zeta_{24}^{c-k}\sqrt{-i\tau} \text{ for all } 1 \le c \le n-1. \tag{9}$$

Using this information we have

Theorem 1.3.4. *For $n = 3, 5, 7$ and 13*

$$\mathfrak{m}_{n,c}(-1/\tau) = \mathfrak{m}_{n,k}(\tau) \text{ for all } 1 \le c \le n-1$$

where $k \equiv -1/c \pmod n$ and $1 \le k \le n-1$.

Proof: For these four primes, (9) in combination with (6) yields

$$\mathfrak{m}_{n,c}(-1/\tau) = \zeta_{24}^{n-c-1}\frac{\eta\left(\frac{-1/\tau+c}{n}\right)}{\eta(-1/\tau)} = \zeta_{24}^{(n-c-1)+(c-k)}\frac{\eta\left(\frac{\tau+k}{n}\right)}{\eta(\tau)} = \mathfrak{m}_{n,k}(\tau).$$

□

1.4. Modular Functions of Level n.

From the theorems above we see that the only roots of unity which appear in the transformations of the level $n = 3, 5, 7$, and 13 functions are ζ_{24}^{n-1} and ζ_{24}^{1-n}. Thus if we let $s(n)$ be as in (5), which for the given values of n becomes

$$s(n) = 24/(n-1), \tag{10}$$

we see that, for a fixed n, an arbitrary modular transformation simply permutes the functions $\mathfrak{m}_{n,c}^{s(n)}$.

In fact we can generalize this result for all odd primes values n, using of course the value of $s(n)$ given originally in (5). This we can do by making use of our original expression for $\mathrm{E}(A;\tau)$ in (7) above. Using the same argument as for the proof of Theorem 1.3.4 we now find that for a general odd prime n,

$$\mathfrak{m}_{n,c}(-1/\tau) = \left(\frac{-k}{n}\right) i^{\frac{1-n}{2}}\zeta_{24}^{(n-1)(c-k)}\,\mathfrak{m}_{n,k}(\tau), \tag{11}$$

where again $k \equiv -1/c \pmod n$.

Now in combination with Theorem 1.3.1 (which applied for a general odd prime n) we have immediately the following:

Theorem 1.4.1. *The functions $\mathfrak{m}_{n,c}^{s(n)}$, for an odd prime n, are permuted up to sign by general modular transformations, where $s(n)$ is as defined in (5).*

In fact it is easy to see that the only time that a minus sign appears is when $n \equiv 1 \pmod 8$ and k is not a square modulo n.

We can now prove

Theorem 1.4.2. *For odd primes n not congruent to 1 modulo 8 the functions $m_{n,c} = \mathfrak{m}_{n,c}^{s(n)}$ are modular functions of level n and if we define $h_n = \mathfrak{m}_{n,0}^{s(n)}$, then for $n = 3, 5, 7$ and 13, h_n is a Hauptmodul for $\Gamma^0(n)$. Similarly for odd primes $n \equiv 1 \pmod 8$ the functions $m_{n,c} = \mathfrak{m}_{n,c}^{2s(n)}$ are modular of level n.*

Proof: From the previous theorem, it is easy to see that the action on the functions $m_{n,c}$ is the same as for a quotient of the Δ function, which is known to be invariant under $\Gamma^0(n)$.

Since $\Gamma^0(n)$ is a congruence subgroup of level n it is clear that h_n is a modular function of level n.

For an odd prime n, the number of cosets of $\Gamma^0(n)$ in the full modular group is $n+1$. As there are precisely $n+1$ functions in the orbit of h_n under the action of the modular group, the precise invariance group of h_n must be $\Gamma^0(n)$.

Since $\Gamma^0(n)$ is a genus zero group for $n = 3$, 5, 7 and 13, it has a Hauptmodul. If, as is customary, we demand the Hauptmodul be chosen so that it has a pole only at $\tau = i\infty$, then given that it will certainly be invariant under the transformation $\tau \to \tau + n$, its q-series must begin with some power of $q^{-1/n}$. However for the primes listed, the function h_n has q-series starting with $q^{-1/n}$ and thus it must be expressible as a polynomial of degree one in the Hauptmodul. In other words, h_n is itself a Hauptmodul.

Finally since each of the other functions $m_{n,c}$, for a fixed n, is a transform of h_n, they belong precisely to conjugates of the group $\Gamma^0(n)$ in the modular group. E.g. for odd primes n not congruent to 1 modulo 8, $\mathfrak{m}_{n,\infty}^{s(n)}$ is a function for $\Gamma_0(n)$. Also, each of these conjugate groups is a level n congruence subgroup, hence our result.

□

Now we wish to prove the identities contained in Theorems 1.1.3, 1.2.3 and 1.2.4. We will also prove generalisations of the first two of these theorems and of the identity of Theorem 1.1.4.

The proof of the three identities is similar in each case. We note that by Theorem 1.4.1 and the comments which follow, the expression on the left hand side in Theorem 1.2.4 is a modular function for the full modular group. The same applies in Theorems 1.1.3 and 1.2.3, except that we first must raise to an appropriate power. Since the left hand sides are then all functions for the full modular group, they lie in $\mathbb{C}(j)$.

Our aim will be to show that in fact the left hand sides in each case are in fact all constants. In the case of Theorems 1.1.3 and 1.2.3, where we have raised to a power, this will be sufficient to show that the products themselves are constants.

But in the products (raised to an appropriate power) we note that since the eta function has no zeroes or poles in the complex upper half plane, then these expressions have none either. Thus the expressions are indeed constants.

By examining the q-series of the actual products themselves one finds that the constants are the ones given in the respective identities. In fact we have the following more general theorem.

Theorem 1.4.3. *For an odd prime n we have*

$$\mathfrak{m}_{n,\infty} \cdot \prod_{c=0}^{n-1} \mathfrak{m}_{n,c} = \zeta_{24}^{\frac{(n-1)^2}{2}} \sqrt{n}.$$

Proof: By the above we merely need to compute the q-series of the product. The leading term of $\mathfrak{m}_{n,c}$ for $1 \le c \le n-1$ is $\zeta_{24}^{n-c-1+\frac{c}{n}} q^{\frac{1-n}{24n}}$. That of $\mathfrak{m}_{n,0}$ is $q^{\frac{1-n}{24n}}$ and that of $\mathfrak{m}_{n,\infty}$ is $\sqrt{n}\, q^{\frac{n-1}{24}}$.

The product is as given in the theorem. □

For the sum of powers of our functions in Theorem 1.2.4 we note that, since the eta function has no zeroes or poles in the complex upper half plane, the expression in question has no poles in this region. Thus from what we already showed above, it is a modular function lying in $\mathbb{C}[j]$.

In the following theorem, we will give a proof, which also holds for the current situation, that this expression is actually a constant. Taking this to be so for now, we simply look at the q-series of this expression and find that the constant is the one given.

The generalisation of Theorem 1.1.4 that we wish to prove is as follows.

Theorem 1.4.4. *For n a prime congruent to 3 modulo 4 the following identity holds.*

$$m_{n,\infty}^{s(n)/2} - m_{n,0}^{s(n)/2} + \sum_{i=1}^{n-1} (-1)^i \, m_{n,i}^{s(n)/2} = 0. \tag{12}$$

Proof: Firstly we will show that the square of the expression in question (the left hand side of the identity in the theorem, which we will denote M for convenience) is a modular function for the full modular group and thus in $\mathbb{C}(j)$. Note that from Theorem 1.4.1, the $s(n)/2$-th powers of our functions are only permuted up to sign. Thus the precise signs of terms in M become relevant.

Firstly it is easy to see from Theorem 1.3.1 that M changes sign under the transformation $\tau \to \tau + 1$. Thus M^2 remains unchanged under this transformation.

From Theorem 1.3.2 the first two terms of the expression M are swapped by the transformation $\tau \to -1/\tau$ except for a change of sign.

Now we must see what happens to the remaining terms of M, which are contained in the sum. It is immediate from (11) that

$$\mathfrak{m}_{n,c}^{s(n)/2}(-1/\tau) = (-1)^{c-k-1} \mathfrak{m}_{n,k}^{s(n)/2}(\tau).$$

Thus the remaining terms of M are also permuted by $\tau \to -1/\tau$, except for a change of sign throughout, and so indeed, M^2 is invariant under this transformation.

Since the function M^2 is invariant under the transformation $\tau \to \tau+1$, the q-series of M^2 must have integer exponents. A simple examination of the leading term of M^2 shows therefore that it's leading term must be the constant term. But then M^2, and hence M, is constant. Thus if the constant term of the q-series of M is 0 then the result is proved.

Firstly we note that the constant term of the q-series of the first term of M is zero, since it only has positive powers of q in its q-expansion. We will now show that the constant term of the q-series of each of the other terms in M is the same, and zero.

Firstly we note that

$$\mathfrak{m}_{n,k} = \zeta_{24}^{n-k-1} \, \frac{\eta((\tau+k)/n)}{\eta(\tau)} = \zeta_{24}^{n-1} \, \frac{\eta((\tau+k)/n)}{\eta(\tau+k)},$$

thus

$$\mathfrak{m}_{n,k}^{s(n)/2} = -\frac{\eta((\tau+k)/n)^{s(n)/2}}{\eta(\tau+k)}.$$

However, this last expression can be obtained from $-[\eta(\tau/n)/\eta(\tau)]^{s(n)/2}$ by applying the transformation $\tau \to \tau + k$ to it. However such a transformation does not change the constant term of the q-series and so the problem is reduced to finding the constant term of this last expression.

If we apply the transformation $\tau \to n\tau$, again the constant term will not change, and it is now sufficient to show that the constant term of the q-series of $W = [\eta(\tau)/\eta(n\tau)]^{s(n)/2}$ is zero, to complete the proof.

But this q-series is given by

$$W = q^{-\frac{s(n)(n-1)}{48}} \cdot \left(\frac{\prod_{i=1}^{\infty}(1-q^i)}{\prod_{i=1}^{\infty}(1-q^{ni})} \right)^{s(n)/2}.$$

However it is clear that the large right hand factor is a q-series in integral powers of q, whilst the power of q on the left is a half integral power, when $n \equiv 3 \pmod 4$. This shows that the q-series of the whole expression has zero constant term, completing the proof of the result stated. □

We have no generalisation of Theorem 1.2.4 at this stage, since, for $n \equiv 1 \pmod 4$ such a generalisation would have to deal with a variety of different rational integer constants which must appear on the right hand sides of the identities. We have computed a large number of these constants and have been unable to determine all their properties.

2. General Construction of Schläfli Modular Equations

From the introduction, the first step in constructing Schläfli modular equations for level n functions is to construct a set of functions whose product is invariant under $\Gamma^0(n)$. We will construct such a function from various transforms of the basic function

$$\mathfrak{m}_{n,0}(\tau) = \frac{\eta(\tau/n)}{\eta(\tau)}.$$

For a fixed n we will denote this basic function by $u(\tau)$. If we wish to emphasize the level n we will write $u_n(\tau)$.

2.1. Linear Transformations of the Function u.

Firstly it is convenient to have a linear transformation rule for $u(\tau)$ as for the eta function. This is derived from the latter which was codified already in Theorem 1.3.3. We have

Theorem 2.1.1. *Let* $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^0(n)$ *be such that* δ *is odd, then*

$$u_n(A\tau) = \left(\frac{n}{|\delta|}\right) \nu_n(\alpha,\beta,\gamma,\delta)\, u_n(\tau), \tag{13}$$

and if δ *is even, then*

$$u_n(A\tau) = \left(\frac{n}{|\delta - \gamma n|}\right) i^{\frac{3(n-1)}{2}} \nu_n(\alpha,\beta,\gamma,\delta)\, u_n(\tau), \tag{14}$$

where

$$\nu_n(\alpha,\beta,\gamma,\delta) = \zeta_{24}^{(n-1)[\delta(\beta/n+\gamma)+(\delta^2-1)\alpha\gamma]}. \tag{15}$$

Proof: Note that

$$\frac{1}{n}\frac{\alpha\tau+\beta}{\gamma\tau+\delta} = \frac{(\alpha/n)(n\tau)+\beta}{\gamma(n\tau)+(n\delta)} = \frac{\alpha(\tau/n)+(\beta/n)}{(n\gamma)(\tau/n)+\delta} \tag{16}$$

Now suppose that δ is odd and positive. Since $n|\beta$, then from the second of the expressions for $\frac{1}{n}A\tau$ in (16) and Lemma (1.3.3) we have

$$\begin{aligned} u_n(A\tau) &= \frac{E\left(\begin{pmatrix} \alpha & \beta/n \\ n\gamma & \delta \end{pmatrix}; \tau/n\right)}{E\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \tau\right)} \frac{\eta(\tau/n)}{\eta(\tau)} \\ &= \left(\frac{n}{\delta}\right) \zeta_{24}^{(n-1)[\delta(\beta/n+\gamma)+(\delta^2-1)\gamma\alpha]}\, u_n(\tau). \end{aligned}$$

If δ is negative, we can multiply $\alpha,\beta,\gamma,\delta$ by -1. Then A represents the same fractional linear transformation, but δ is now positive. This observation leads to the stated result.

Now suppose that δ is even. The problem here is that the transformation formula for the eta function, which we quoted, does not allow for δ to be even. Note that this case only occurs if n is odd, for if n is even, then since $n \mid \beta$ and $\alpha\delta - \beta\gamma = 1$ we have that δ is always odd in this case.

The way we deal with δ being even, is to decompose the transformation A into two other transformations with odd lower right entries:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta - \alpha n \\ \gamma & \delta - \gamma n \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

We multiplying the entries of the first matrix on the right by -1 if necessary, so that $\delta - \gamma n$ is positive. Then we apply the first part of the theorem, which we have already proved, twice, and

$$\begin{aligned} u_n(A\tau) &= \left(\frac{n}{|\delta - \gamma n|}\right) \zeta_{24}^{(n-1)[(\delta-\gamma n)((\beta-\alpha n)/n+\gamma)+((\delta-\gamma n)^2-1)\alpha\gamma+1]} u_n(\tau) \\ &= \left(\frac{n}{|\delta - \gamma n|}\right) \nu_n(\alpha,\beta,\gamma,\delta)\, \zeta_{24}^{(n-1)[1-\gamma\beta-\alpha\delta+n\alpha\gamma-n\gamma^2-2n\alpha\gamma^2\delta+n^2\alpha\gamma^3]}. \end{aligned}$$

It is clear we only need to consider the exponent of ζ_{24} modulo 24.

Firstly we look at the exponent modulo 3. There are three cases. If $n \equiv 0 \pmod 3$ then we write $1 - \alpha\delta$ as $-\beta\gamma$ and note that $3 \mid n \mid \beta$.

If $n \equiv 1 \pmod 3$ then we observe that the exponent has a factor of $(n-1)$.

If $n \equiv 2 \pmod 3$ then again writing $1 - \alpha\delta$ as $-\beta\gamma$ we find that there is a factor of $\gamma(1-\gamma^2)$ in the exponent, which is always divisible by 3.

Thus in all cases the exponent is divisible by 3.

Now we work modulo 8. Since $(n-1)$ is even, we need only look at the rest of the exponent modulo 4. Given that δ is even, n and γ are odd and $8 \mid (n-1)(n+1)$, the congruence is easy to evaluate, and we find that the remainder of the exponent is $2 - n$ modulo 4. Now considering the entire exponent modulo 8 and recalling that n is odd, we obtain the result stated.

$\square$

Note that ν_n is a root of unity, and in fact it is, at worst, an $s(n)$-th root of unity with $s(n)$ defined as above, (5).

In this paper, we will find explict modular equations for the following levels:

$$n = 2, 3, 5, 7, 11, 13, 17. \tag{17}$$

We make this restriction only for reasons of time. The method continues to work for other prime values of n.

Theorem 2.1.2. *For the values of n that we are interested in,* (17)*, $u_n^{s(n)}$ is invariant under $\Gamma^0(n)$, except when $n = 17$, where we require $u_n^{2s(n)}$.*

Proof: Firstly it is easy to see that there exists a set of generators for $\Gamma^0(n)$ with the lower right entry δ always odd. For, if n is odd, take an arbitrary set of generators including T^n, then any generator which has lower right entry even, can be composed with T^n to give one which has odd lower right entry. If n is even, it is trivially the case that δ is always odd.

For even n, $s(n)$ is always even and so Jacobi symbols don't affect matters.

Now for odd n we are almost there except for the Jacobi symbol which appears in (13). Clearly when $s(n)$ is divisible by 2, the Jacobi symbols are irrelevant. But this is the case except when $n \equiv 1 \pmod 8$. But the only such values n in our list are either squares, or the exceptional case $n = 17$. For the squares, the Jacobi symbols are trivial.

□

2.2. **Functions Permuted by $\Gamma^0(n)$.** We wish to describe a set of functions permuted by $\Gamma^0(n)$ as prescribed by the method outlined in the introduction. The basic building blocks in this process will be the following functions.

Definition 2.2.1. *For a fixed n we define the following set of functions in terms of the function u_n, described at the start of this section,*

$$u(\tau) = u_n(\tau), \quad v_{a,b,d}(\tau) = u_n\left(\frac{a\tau + b}{d}\right); \;\; \textit{for} \;\; (a,b,d) = 1,\, ad = m,$$

and b traversing a complete set of residues modulo d (which we fix in the sequel).

In looking for functions which are permuted under the action of $\Gamma^0(n)$ we will consider as candidates, quotients of functions of the form $v_{a,b,d}$ by an appropriate power of the function u, depending only on m and n. To do this, we need to see how the $v_{a,b,d}$ transform in comparison to various powers of u.

Let us consider the simplest situation, namely where $(m, s(n)) = 1$. The results in this case are given in the following theorem.

Theorem 2.2.2. *Let m be given, with $(m,n) = 1$ and $(m, s(n)) = 1$ and let m' be the smallest positive integer (in the case of $n = 17$ take the smallest* odd *positive integer) such that $m' \equiv m \pmod{s(n)}$, then the functions*

$$w_{a,b,d} = \left(\frac{n}{d}\right)\frac{v_{a,b,d}}{u^{m'}}$$

are permuted by $\tau \to A\tau$ where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^0(n)$ and δ is odd.

Proof: The appropriate method in this case is to demand that the representatives b modulo d, in the definition of our functions $v_{a,b,d}$, are always chosen such that $n \cdot s(n) \,|\, b$. This we can always do, since if $(m, s(n)) = 1$ then $(d, s(n)) = 1$, and we also have $(m,n) = 1$.

Now we can easily find the action of $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^0(n)$ on our functions $v_{a,b,d}$. We simply consider the following matrix equation:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix},$$

with $a'd' = m$, $n \cdot s(n) \,|\, b'$ and where the first matrix on the right hand side (which we will call A') is also in $\Gamma^0(n)$. It is not difficult to show that such a matrix equation is always soluble.

This matrix equation can be read as follows. Applying a transformation $\tau \to A\tau$ to the function $v_{a,b,d}$, yields, up to a linear transformation $A' \in \Gamma^0(n)$, one of our other functions $v_{a',b',d'}$.

Modulo $n \cdot s(n)$, the first and last matrices in the matrix equation are diagonal, and it is easy to see that certainly

$$\alpha' \equiv \frac{a}{a'}\alpha, \quad \beta'/n \equiv \frac{a}{d'}\beta/n, \quad \gamma' \equiv \frac{d}{a'}\gamma \;\text{ and }\; \delta' \equiv \frac{d}{d'}\delta \pmod{s(n)}. \tag{18}$$

Thus by making use of the transformation laws given by Theorem 2.1.1 we can now determine the roots of unity which result from the linear transformation A'.

Firstly the $n-1$ in the exponent of $\nu(\alpha', \beta', \gamma', \delta')$ ensures we only have to look at the remainder of the exponent modulo $s(n)$.

We make use of the following fact. Since $s(n)$ is a factor of 24, then for any integer q with $(q, s(n)) = 1$ we have $q^2 \equiv 1 \pmod{s(n)}$, i.e. $q \equiv 1/q \pmod{s(n)}$.

In particular, since $ad = a'd' = m$ and $(m, s(n)) = 1$, the congruences we derived above enable us to establish the following equality from the definition of ν:

$$\nu(\alpha', \beta', \gamma', \delta') = \nu(\alpha, \beta, \gamma, \delta)^{m'},$$

where $m' \equiv m \pmod{s(n)}$.

Now it may be that δ' is not odd. However we claim this can only happen if $n \equiv 1 \pmod 8$, in which case the factor $i^{\frac{3(n-1)}{2}}$, which would be required in Theorem 2.1.1, would be trivial. For, in all other cases $s(n)$ is even and since $(m, s(n)) = 1$ we must have m odd. But then d and d' are odd, and since $2 \mid s(n) \mid b'$ then the relation

$$d\delta = \gamma' b' + \delta' d'. \tag{19}$$

from the matrix equation above, reveals that δ' is odd, as δ is.

Now we must look at the Jacobi symbols which can occur in Theorem 2.1.1.

Firstly note that from (19), $d'\delta' \equiv d\delta \pmod n$.

We'll start with the case where δ' is odd. If n is odd then since $d'|\delta'|$ is also odd and since it is clear that the two values are coprime, then we have by the reciprocity law for Jacobi symbols, and the congruence just stated

$$\begin{aligned}\left(\frac{n}{|d'\delta'|}\right) &= (-1)^{\frac{(n-1)|d'\delta'|-1)}{4}} \left(\frac{|d'\delta'|}{n}\right) \\ &= (-1)^{\frac{(n-1)(|d'\delta'|-1)}{4}} \left(\frac{|d\delta|}{n}\right) = (-1)^{\frac{(n-1)(|d'\delta'|-|d\delta|)}{4}} \left(\frac{n}{|d\delta|}\right).\end{aligned} \tag{20}$$

The sign before the final Jacobi symbol is always $+1$, for, since $8 \mid (n-1) \cdot s(n)$ and $s(n) \mid b'$ we have from (19) that

$$(n-1)|d'\delta'| \equiv (n-1)|d\delta| \pmod 8.$$

For n even, a relationship similar to (20) still holds, for we can first derive a similar expression involving n' the largest odd factor of n and note that $8 \mid s(n)$ so that the sign before the Jacobi symbol on the right hand side again is always 1. Finally we note that $\left(\frac{2}{|d'\delta'|}\right) = \left(\frac{2}{|d\delta|}\right)$ by virtue of (19) since $8 \mid s(n) \mid b'$.

This analysis is sufficient to complete the proof in the case where δ' is odd.

Now we turn to the case where δ' is even, and hence $n \equiv 1 \pmod 8$ as shown above. When n is a square, the Jacobi symbols involved are all trivial and the result follows immediately.

For the remainder we find that for the Jacobi symbol appearing in the relevant transformation law,

$$\left(\frac{n}{|\delta' - \gamma' n|}\right) = \left(\frac{|\delta' - \gamma' n|}{n}\right) = \left(\frac{|\delta'|}{n}\right) = \left(\frac{n}{|\delta'|}\right), \tag{21}$$

if we allow that the final symbol is actually a Kronecker symbol. This follows since $n \equiv 1 \pmod 8$, making the relevant reciprocity law and the symbol $\left(\frac{n}{2}\right)$, trivial.

It is also easy to see that

$$\left(\frac{n}{d'|\delta'|}\right) = \left(\frac{n}{d|\delta|}\right)$$

for similar reasons. Again this equality leads to the result as stated in the theorem.

□

Now we come to the more complex case where $(m, s(n)) \neq 1$. We can no longer work modulo $n \cdot s(n)$ as we did to obtain (18). Of course we still have $(n, m) = 1$, but we can no longer demand that b and b' be divisible by $s(n)$, since d and d' are not necessarily coprime to $s(n)$.

The basic method for dealing with this problem is to raise our functions $w_{a,b,d}$ to some carefully chosen power. This has the effect of raising the roots of unity we are dealing with to the same power, thereby making the congruences we have to deal with much simpler.

For example if $s(n)$ is divisible by 3 then raising our functions $w_{a,b,d}$ to the third power means that we only have to deal with congruences modulo $n \cdot s(n)/3$.

Let $\mathrm{ord}_2 s(n)$ be the highest power of 2 which divides $s(n)$, etc. It is almost sufficient to raise our functions to the power ρ defined as follows:

$$\rho = \begin{cases} 2^{\mathrm{ord}_2 s(n)}, & 2 \mid (s(n), m),\ 3 \nmid (s(n), m) \\ 3, & (s(n), m) = 3 \\ 3 \cdot 2^{\mathrm{ord}_2 s(n)}, & 6 \mid (s(n), m). \end{cases} \tag{22}$$

Note that these are the only cases, since $s(n) \mid 24$ in all cases.

Raising to this power ρ effectively ends the dependence of our congruences on factors which $s(n)$ and m have in common and in most cases we will raise to this power. However there is an exception to this rule. This occurs in the case where 2 is a common factor of $s(n)$ and m.

This exceptional cases corresponds to the first case of the following theorem, where we suggest raising to the power $\rho/2$. This of course only ensures that our functions $w_{a,b,d}$ are permuted up to sign. However later we will deal with this problem by introducing some additional functions to our collection, which each turn out to be -1 times one of the original functions $w_{a,b,d}$.

Theorem 2.2.3. *(i) Suppose $(m, n) = 1$, $2 \mid (m, s(n))$ and ρ is defined as above. Letting m' be the smallest positive residue of m modulo $s(n)/\rho$ then the functions*

$$w_{a,b,d} = \left(\frac{v_{a,b,d}}{u^{m'}}\right)^{\rho/2} \tag{23}$$

with $ad = m$ and b traversing a complete set of residues modulo d (chosen such that $n \cdot s(n)/\rho \mid b$), are permuted up to sign by all elements of $\Gamma^0(n)$ with odd lower right entries.

(ii) Suppose $(m, n) = 1$ and $(m, s(n)) = 3$. Now $\rho = 3$, and so letting m' be the smallest positive residue of m modulo $s(n)/3$ (except for $n = 17$ where we take the smallest such value m' which is odd) we have that the functions

$$w_{a,b,d} = \left(\left(\frac{n}{d}\right) \frac{v_{a,b,d}}{u^{m'}}\right)^3, \tag{24}$$

are permuted by the action of all elements of $\Gamma^0(n)$ with odd lower right entries.

Proof: The proof uses essentially the same argument as that of the last theorem except that we work modulo $n \cdot s(n)/\rho$ and compensate by raising to the power ρ for full permutation and to the power $\rho/2$ for permutation up to sign.

However we need to take any Jacobi symbols and factors of $i^{\frac{3(n-1)}{2}}$ that occur in the transformation laws into account. The breakdown is as follows.

We start with the case where n is odd. In the case where we raise to the power $\rho/2$ we are only interested in permutation up to sign, and so Jacobi symbols are not relevant here.

In the remaining cases, $(m, s(n)) = 3$, and we are simply raising to a power $\rho = 3$. This means that it is still the case that $8 \,|\, (n-1) \cdot s(n)/\rho \,|\, b'$. Thus the argument from the previous theorem for n odd, still holds, so long as d' is odd.

However it may now be the case that $2 \,|\, m$ in which case this last condition no longer holds. However this complication can only occur if $2 \not| \; s(n)$. But then $n \equiv 1 \pmod 8$ and the symbols in (20) are all trivial anyway (if we think of them as Kronecker symbols) and so the result is the same.

For the case where n is even, $(m, 2) = 1$, and so the argument of the previous theorem, appropriately modified, is also sufficient.

Now we deal with any factors of $\epsilon = i^{\frac{3(n-1)}{2}}$ that may arise when δ' is even. However the argument of the previous theorem breaks down for this situation only if m and $s(n)$ are allowed to share a factor of 2. However if this is the case then we are at least raising our functions to a power of 2 (making $\epsilon = \pm 1$, which is harmless considering we are only interested in permutation up to sign) unless $2 \,|\, s(n)$ but $4 \not| \; s(n)$. However this can only be the case if $n \equiv 5 \pmod 8$. But then $\epsilon = \pm 1$ again. Thus in all cases the factors ϵ are irrelevant.

□

Now we must show how to deal with our functions $w_{a,b,d}$ if they are only permuted up to sign by transformations in $\Gamma^0(n)$.

In these cases, $2 \,|\, (m, s(n))$. Since n must be odd in this case, $s(n)$ can be divisible by 2 precisely once or twice. Letting ρ' be the power of 2 appearing in ρ, these two cases correspond respectively to $\rho' = 2$ and $\rho' = 4$.

To begin with, it is easy to see that $n \cdot s(n)$ is the period of the function $u(\tau)$. Thus, when $2 \,|\, d$, letting $b'' = nd \cdot s(n)/2$ in the first of the two cases just mentioned and $b'' = nd \cdot s(n)/4$ in the second of these two cases, we have that

$$v_{a,b+b'',d}^{\rho/2} = u\left(\frac{a\tau + b + nd \cdot s(n)/\rho'}{d}\right)^{\rho/2} = -u\left(\frac{a\tau + b}{d}\right)^{\rho/2} = -v_{a,b,d}^{\rho/2}. \tag{25}$$

Therefore, when $2 \,|\, d$ we add the extra functions $v_{a,b+b'',d}$ to our list of functions $v_{a,b,d}$.

Now, if the functions $w_{a,b,d} = (v_{a,b,d}/u^{m'})^{\rho/2}$, involving only our *original* functions $v_{a,b,d}$, are permuted up to sign, (more precisely, the sign may change when a function $v_{a,b,d}$ is involved, with $2 \,|\, d$), then the full set of functions $w_{a,b,d} = (v_{a,b,d}/u^{m'})^{\rho/2}$ involving *all* the $v_{a,b,d}$, will actually be fully permuted by the action of $\Gamma^0(n)$.

Note that we could try to replicate this whole argument in the case where 3 (or even 4) is a common factor of m and $s(n)$. Though there is no hinderance to it here, it turns out to be ultimately pointless, since when we try to relate the q-series of the collections of functions $A_{a,b,d}$ and $B_{a,b,d}$, that we will end up defining, we will find that we cannot do so. Therefore this argument for reducing the size of modular equations only works in the case where 2 is a factor of m and $s(n)$.

2.3. **The Functions $P_{a,b,d}$ and $Q_{a,b,d}$.** Now according to the introduction we are required to construct functions (involving our original functions $v_{a,b,d}$) of the form

$$P_{a,b,d} = (uv_{a,b,d})^k \quad \text{and} \quad Q_{a,b,d} = (v_{a,b,d}/u)^l \tag{26}$$

for $k, l \in \mathbb{N}$, which are permuted in precisely the same way by elements of $\Gamma^0(n)$ (or up to sign where applicable, though it must be the same sign in both cases).

We will construct such functions by multiplying or dividing the functions $w_{a,b,d}$ by some power, u^j, of the fundamental function which is invariant under the action of $\Gamma^0(n)$, as per Theorem 2.1.2. Or, if we require only permutation up to sign, we will multiply by some power of $u^{j/2}$ where u^j is the invariant function of Theorem 2.1.2.

Theorem 2.3.1. *The following sets of functions are permuted (or permuted up to sign, as the case may be) by $\Gamma^0(n)$ (with the same signs in both cases for any given transformation in $\Gamma^0(n)$):*

$$P_{a,b,d} = w^k_{a,b,d}\, u^{jt_1}; \;\; t_1 \in \mathbb{Z} \tag{27}$$

$$Q_{a,b,d} = w^l_{a,b,d}\, u^{jt_2}; \;\; t_2 \in \mathbb{Z}, \tag{28}$$

(or, for the first part of Theorem 2.2.3, with j replaced by $j/2$, in these functions), as follows:

Under the conditions of Theorem 2.2.2

$$k = \frac{j}{gcd(m'+1, j)} \;\; \text{and} \;\; l = \frac{j}{gcd(m'-1, j)}. \tag{29}$$

For both parts (i) and (ii) of Theorem 2.2.3 we take

$$k = \frac{j}{gcd(\rho m'+\rho, j)} \;\; \text{and} \;\; l = \frac{j}{gcd(\rho m'-\rho, j)}. \tag{30}$$

Proof: We must deal with each case in Theorem 2.2.2 and in Theorem 2.2.3 separately. For the first of these theorems, we know that the $w_{a,b,d} = v_{a,b,d}/u^{m'}$ are permuted, thus if we let

$$P_{a,b,d} = w^k_{a,b,d}\, u^{jt_1}; \;\; \text{for some } t_1 \in \mathbb{Z}, \tag{31}$$

$$Q_{a,b,d} = w^l_{a,b,d}\, u^{jt_2}; \;\; \text{for some } t_2 \in \mathbb{Z}, \tag{32}$$

then our conditions are satisfied if we let $(m'+1)k = jt_1$ and $(m'-1)l = jt_2$.

We note that these equations are soluble for t_1 and t_2 if we set the values k and l as per the first part of the theorem.

We move on to the Theorem 2.2.3, part (i). Here we only require permutation up to sign, and therefore if u^j is the invariant power of the fundamental function, we are able to make use of $u^{j/2}$ which is invariant up to sign (note j is always divisible by 2 here).

In this case we will require

$$(\rho m'/2 + \rho/2)k = jt_1/2, \quad (\rho m'/2 - \rho/2)l = jt_2/2, \tag{33}$$

leading to k and l as defined, so long as $jt_1/2$ and $jt_2/2$ have the same parity.

This extra parity condition is so that $P_{a,b,d}$ and $Q_{a,b,d}$ continue to permute in exactly the same way with the *same* signs. However, if $j/2$ is even it is clear that the parity is the same in both cases. If $j/2$ is odd, then 2 divides j precisely once and hence also ρ precisely once. But then if we use the k and l as suggested by the theorem, k and l are then odd and we have that the parities of t_1 and t_2 depend only on the respective parities of $m'+1$ and $m'-1$, which are the same. This means that the condition for the parity to be the same always holds if we use k and l as given. This leads to the stated result.

For part (ii) of Theorem 2.2.3, we again require full permutation and we therefore make use of u^j as the invariant power of the fundamental function. The equations we obtain are

$$(3m'+3)k = jt_1, \quad (3m'-3)l = jt_2, \tag{34}$$

with the same result as in the previous case with $\rho = 3$.

□

2.4. **The Fricke Involution.** We not only want functions which have the properties just mentioned but which are also invariant at least up to sign under a Fricke involution, $\tau \to -n/\tau$. To this end we prove

Lemma 2.4.1. *The transformation* $\tau \to -\frac{n}{\tau}$ *sends* u_n *to* $\frac{\sqrt{n}}{u_n}$.

Proof: From the definition,

$$u_n(n\tau) = \frac{\eta(\tau)}{\eta(n\tau)}.$$

Sending $\tau \to -1/\tau$ and applying the transformation formula of the Dedekind eta function to the right hand side, we obtain the required result.

□

We now know what the Fricke involution does to $u = u_n(\tau)$ and we are in a position to see what it does to the $w_{a,b,d}$. We must deal with each of the cases of the Theorems 2.2.2 and 2.2.3 separately.

Theorem 2.4.2. *(i) In the situation of Theorem 2.2.2 we find that the Fricke involution has the following action:*

$$w_{a,b,d}(-n/\tau) = \left(\frac{n}{m}\right)\frac{\sqrt{n}^{1-m'}}{w_{a',b',d'}}.$$

(ii) For part (i) of Theorem 2.2.3 we find,

$$w_{a,b,d}(-n/\tau) = \pm\frac{\left(\sqrt{n}^{1-m'}\right)^{\rho/2}}{w_{a',b',d'}}.$$

(iii) For part (ii) of that theorem we have,

$$w_{a,b,d}(-n/\tau) = \left(\frac{n}{m}\right)\frac{\left(\sqrt{n}^{1-m'}\right)^{3}}{w_{a',b',d'}}.$$

Proof: In all cases we work with the following matrix equation which allows us to determine the action of the Fricke involution $\tau \to -n/\tau$ on a function $v_{a,b,d}$. We then adjust our results by raising to the appropriate powers and take into account Jacobi symbols etc., in order to determine the action on the $w_{a,b,d}$.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & -n \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & -n \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix},$$

where the first matrix on the right is in $\Gamma^0(n)$.

We start by noting that in all cases this matrix equation requires

$$b = \beta a', \quad -na = -\alpha n d' + \beta b', \quad d = \delta a', \quad 0 = -\gamma n d' + \delta b'.$$

Firstly under the hypothesis of Theorem 2.2.2, we have $n \cdot s(n) \mid b,\, b'$ and given that $(m, s(n)) = 1$, it is easy to derive the following congruences:

$$\alpha \equiv a/d', \quad \beta/n \equiv 0 \quad \gamma \equiv 0 \quad \delta \equiv d/a' \pmod{s(n)}.$$

It is easy to see from these congruences that that the value of $\nu(\alpha, \beta, \gamma, \delta)$ is always 1.

If n is even, then β is even and so δ is odd. If n is odd, then unless $n \equiv 1 \pmod 8$, $s(n)$ is even and so m is odd. Thus the third equation above shows that δ is odd.

In the case where $n \equiv 1 \pmod 8$ any transformation factor $i^{\frac{3(n-1)}{2}}$ is trivial. If δ is even, then the Jacobi symbol associated with the transformation is $\left(\frac{n}{|\delta - \gamma n|}\right)$. By an argument as per (21) we see that this Jacobi symbol is the same as $\left(\frac{n}{\delta}\right)$ thought of as a Kronecker symbol.

Now in all cases we have from the third equation above, that

$$\left(\frac{n}{\delta}\right) = \left(\frac{n}{d}\right)\left(\frac{n}{a'}\right) = \left(\frac{n}{d}\right)\left(\frac{n}{d'}\right)\left(\frac{n}{m}\right).$$

Thus in all cases we find (by reference to the previous lemma) the result as given by the first part of the theorem.

For the case (i) of Theorem 2.2.3 we will not care which sign the Fricke involution induces, so again we do not analyse Jacobi symbols. As usual with this case, the only obstacle to the prior argument going through is that 2 or 3 may be a common factor of m and $s(n)$. However, if 3 is such a common factor, then $\rho/2$ is divisible by 3 and raising to this power in the definition of $w_{a,b,d}$ makes the evaluation of the exponent of $\nu(\alpha, \beta, \gamma, \delta)$, modulo 3, irrelevant. If 2 is such a common factor we end up raising our functions to precisely the right power to give us a factor of ± 1 when we evaluate $\nu(\alpha, \beta, \gamma, \delta)$. Since we are only interested in the action of the Fricke involution up to sign we have the result stated.

The situation of case (ii) of Theorem 2.2.3 causes no new difficulties and we find precisely the result as stated.

□

2.5. **The Functions $A_{a,b,d}$ and $B_{a,b,d}$.** As can be seen from the results of the previous subsection, in some cases the sign the Fricke involution induces is irrelevant, and in all other cases it is given by $\left(\frac{n}{m}\right)$.

It seems logical to define functions in the latter cases as follows:

$$A_{a,b,d} = P_{a,b,d} + \left(n\left(\frac{n}{m}\right)\right)^k / P_{a,b,d} \quad \text{and} \quad B_{a,b,d} = Q_{a,b,d} + \left(\frac{n}{m}\right)^l / Q_{a,b,d}.$$

With this definition, the sets of functions are permuted by the Fricke involution and by elements of $\Gamma^0(n)$. However unfortunately there are some problems with this simplistic definition. These manifest themselves in cases (i) and (iii) of the above theorem when $\left(\frac{n}{m}\right) = -1$ and k is odd.

To understand the problem in this situation, we turn to the requirements (vi) and (vii) of the introduction. We begin by supposing that a polynomial with rational integer coefficients $F(X, Y)$ has been found such that the q-series of $F(A_\infty, B_\infty) = F(A_{m,0,1}, B_{m,0,1})$ vanishes at least up to and including the constant term (and when m is composite, up to an even higher power of q as required).

It is not hard to see that the transformation $\tau \to \tau/m$ takes the function $uv_{m,0,1}$ to $uv_{1,0,m}$ and $u/v_{m,0,1}$ to $v_{1,0,m}/u$. However noting carefully the definitions, we see that this takes $A_{m,0,1}$ to $-A_{1,0,m}$ and $B_{m,0,1}$ to $B_{1,0,m}$.

Now consider what happens to the q-series of one of our functions under such a transformation. The nome q is simply replaced by $q^{1/m}$. Thus if the q-series of one of the functions vanishes up to and including the constant term (and possibly further) then the q-series of the transformed function also vanishes up to and including the constant term (and possibly further).

So we have shown that the q-series of $B_{m,0,1}$ is related to that of $B_{1,0,m}$ but the q-series of $A_{m,0,1}$ is related to that of $-A_{1,0,m}$. This is a violation of conditions (vi) and (vii) of the introduction.

We will apply the following solution to this problem:

Solution : We change the signs used in the definition of $A_{a,b,d}$ and $B_{a,b,d}$. This causes the problematic transformation mentioned above to send $A_{m,0,1}$ to $-A_{1,0,m}$ and $B_{m,0,1}$ to $-B_{1,0,m}$. Now we simply choose our polynomial $F(X,Y)$ such that the total degree of each monomial has the same parity. Note that now the Fricke involution only permutes the $F_{a,b,d}$ up to sign, but the conditions as set out in the introduction are still met.

It is easy to check that (even after applying the solution above) successive transformations of the form $\tau \to \tau + n \cdot s(n)/\rho$, (where $\rho = 1$ for case (i) of the above theorem), take the functions $A_{1,0,m}$ and $B_{1,0,m}$ to each of the other functions of the form $A_{1,b,m}$ and $B_{1,b,m}$ respectively, in turn. These transformations also relate the q-series in the manner required by the introduction.

Obviously this whole argument follows even more easily for case (ii) of the above theorem since the sign change issues are not relevant here, since we can just add in the extra functions $w_{a,b,d}$ as outlined previously for this case.

It now remains only to use the method we have developed to construct explicitly some modular equations. We recall that the aim is to make sure the q-series of $F_{m,0,1} = F(A_{m,0,1}, B_{m,0,1})$ vanishes to a high enough degree to ensure the vanishing of the q-series of $G = \prod_{a,b,d} F_{a,b,d}$ up to and including the constant term.

We note that if m is prime then all the functions $F_{a,b,d}$ are of the form $F_{m,0,1}$ or $F_{1,b,m}$ for some b. These are all permuted by the transformations mentioned above. Thus in this case, (m prime), it is sufficient to have the q-series of $F_{m,0,1}$ disappear up to and including the constant term.

However when m is not prime, this is no longer the case. We must individually calculate the leading q-terms of the other functions $F_{a,b,d}$ (which don't vanish identically), where $a \neq m \neq d$, and determine up to which point the q-series of $F_{m,0,1}$ must vanish in order to cancel out the contribution from these spurious ones. Coefficients of the q-series are not important here, only a lower bound on the leading q-power in each case.

We start by noting that the function $u_n(\tau)$ begins $q^{\frac{1-n}{24n}} + \cdots$. Thus up to some leading coefficient, $v_{a,b,d}/u$ begins $q^{\frac{(a-d)(1-n)}{24nd}} + \cdots$.

We are firstly interested in the most negative q-power which appears in $A_{a,b,d}$ and $B_{a,b,d}$. Writing $q_n = q^{\frac{(1-n)}{24n}}$ these powers are respectively $q_n^{|a-d|k/d}$ and $q_n^{|a-d|l/d}$.

Consider now the q-series of a monomial $A^\alpha B^\beta$ of the polynomial $F(A,B)$. Its leading q-power is given by $q_n^{|a-d|(\alpha k+\beta l)/d}$. But k and l do not depend on a,b,d, thus it is easy to see that for each a,b,d the monomial having the lowest q-power is always the same monomial in each case. To save superscripts, let us just denote this 'worst' monomial $A^\alpha B^\beta$.

Now we are able to determine the complete contribution to the q-series of G of the spurious $F_{a,b,d}$. For each factor a of n not equal to 1 or n, the corresponding d is fixed ($ad = m$). Now b can run over a complete set of classes modulo d, and there are at most d of these. Thus for a fixed $a \,|\, n$ there are at most d functions

$F_{a,b,d}$. The contribution of their worst q-powers to the q-series of G is thus at most $q_n^{|a-d|(\alpha k+\beta l)}$. Thus the total contribution of all the spurious $F_{a,b,d}$ is at worst

$$q_n^{(\alpha k+\beta l)\Sigma} \text{ where } \Sigma = \sum_{a|m} |a - m/a| - 2(m-1).$$

In the sequel we will compute examples for composite m only for the cases $m = 4, 6, 8, 9, 10, 15$. In all these cases it is easy to see that the total contribution to the q-series of G from the spurious $F_{a,b,d}$ does not come close to exceeding the worst power of q which appears amongst the monomials of $F_{m,0,1}$. Thus it is clear that if we wish to compensate for the effect of the spurious $F_{a,b,d}$ we simply need to make the q-series of $F_{m,0,1}$ disappear up to a positive power of q equal in magnitude to the worst negative power of q coming from its worst monomial.

We picked this condition to work with since it is easy to verify in practise. One always at some point computes the q-series of the worst monomial appearing in $F_{m,0,1}$ and one can then set the overall q-series precision accordingly. In practise we worked with much higher precision in the q-series than was necessary.

Curiously, once we had found an $F_{m,0,1}$ whose q-series vanished even one place beyond the constant term, we invariably had it vanish identically. This clearly implies that the bound we give here is not in any sense tight. However we see no easy way of lowering it at this stage.

In the general case, where m is not one of the values listed above, it is easy to see how to compute the necessary q-series precision. Since in general the sum of divisors of m is not large compared with m, the bound we have given is also always quite practical and is easy to compute.

3. New Modular Equations for the Weber Functions

The first non-trivial case we can consider is $n = 2$. This corresponds to Schläfli modular equations for the ordinary Weber function $\mathfrak{f}_1(\tau)$.

Since Weber has obtained such modular equations for small prime values of m we will restrict our computations to composite values, which Weber is silent on. However, since we must have $(m, n) = 1$, this suggests we should look at the values $m = 9$ and 15 as examples of our method.

We have that $s(n)$ as given by (5) is 24 and thus in both of the cases we are interested in, $(m, s(n)) = 3$. We are thus in the situation of the second case of Theorem 2.2.3. Thus $\rho = 3$ and

$$w_{a,b,d} = \left(\frac{v_{a,b,d}}{u^{m'}}\right)^3.$$

Now working through the definitions as set out by Theorem 2.3.1 we find the following values must be taken for $P_{a,b,d}$ and $Q_{a,b,d}$.

m	$P_{a,b,d}$	$Q_{a,b,d}$
9	$(u \cdot v_{a,b,d})^6$	$(v_{a,b,d}/u)^3$
15	$(u \cdot v_{a,b,d})^3$	$(v_{a,b,d}/u)^6$

In both cases sign changes from the Fricke involution are irrelevant so we set the sign to be $+1$ in the definition of $A_{a,b,d}$ and $B_{a,b,d}$.

We compute the q-series of the functions $A_{m,0,1}$ and $B_{m,0,1}$ (which we denote A and B for simplicity) and we look for a polynomial $F(X, Y) \in \mathbb{Z}[X, Y]$, such that the q-series of $F(A,B)$ vanishes sufficiently far. This can be done easily with a computer algebra package, such as [14].

To avoid fractional exponents, it is convenient to expand A and B in the nome q^2, which does not impact the resulting modular equation. As the q-series are somewhat unenlightening, we supress them here. Examples can be found in the author's thesis [13].

To find a suitable polynomial $F(A,B)$ we start with two monomials in A and B whose leading q-terms are the same, and subtract them. We continue to add monomials to this expression which cancel further powers of q from the q-expansion of our cumulative polynomial, until we cannot progress further (sometimes there is no monomial in A and B, which can be subtracted, to get rid of the leading power of q in the q-series).

If this is not already the polynomial we are looking for, then we start again with a different pair of monomials both having the same leading q-term. Applying the same procedure, we obtain another polynomial whose q-series vanishes up to some point. But this polynomial in A and B will be formally linearly independent of the first.

We keep creating such linearly independent polynomials until we have sufficiently many of them that some linear combination of them will make the powers of q vanish to the required point.

This algorithm results in the modular equations given in the following table.

m	Level 2 Modular Equation
9	$(B-1)A^2 + (-B^6 + 28B^5 - 292B^4 + 1408B^3$ $-3200B^2 + 3072B - 1024) = 0$
15	$A^8 + (B^2-47)A^7 + (-45B^2+657)A^6 + (574B^2-1448)A^5 +$ $(-555B^2 - 19348)A^4 + (45B^4 - 14344B^2 - 6832)A^3$ $+(2895B^4 - 5880B^2 + 300784)A^2 + (39248B^4 + 159104B^2$ $+898304)A + (-B^6 + 141516B^4 + 51664B^2 + 937024) = 0$

4. Schläfli-type Modular Equations for Level Three Functions

The next case to consider is that of modular equations for level three functions, i.e. where $n = 3$. Here $s(n) = 12$, thus we only need to consider values of m modulo 12. Note that $\left(\frac{n}{m}\right)$, for odd m, also only depends on m modulo 12.

Here we find the first case where $\left(\frac{n}{m}\right) = -1$ and k is odd, namely where $m \equiv 7 \pmod{12}$. Thus we need to apply the solution which we mentioned in Section 2.5.

We are only interested in the definitions of the functions $A_{m,0,1}$ and $B_{m,0,1}$. These we give in the following table, in which we omit the subscripts $m,0,1$ for simplicity.

$m \pmod{12}$	A	B
1	$(uv)^6 + 3^6/(uv)^6$	$(v/u) + (u/v)$
2	$(uv)^2 + 3^2/(uv)^2$	$(v/u)^6 + (u/v)^6$
4	$(uv)^6 + 3^6/(uv)^6$	$(v/u)^2 + (u/v)^2$
5	$(uv)^2 + 3^2/(uv)^2$	$(v/u)^3 - (u/v)^3$
7	$(uv)^3 + 3^3/(uv)^3$	$(v/u)^2 - (u/v)^2$
8	$(uv)^2 + 3^2/(uv)^2$	$(v/u)^6 + (u/v)^6$
10	$(uv)^6 + 3^6/(uv)^6$	$(v/u)^2 + (u/v)^2$
11	$(uv) + 3/(uv)$	$(v/u)^6 + (u/v)^6$

After some computation we obtain a smattering of sample modular equations. Clearly they are more complex when the degree is composite, so we list fewer of these.

m	Level 3 Modular Equation
2	$A=B$
4	$AB-B^6+21B^4-93B^2-8=0$
5	$A=B-5$
7	$A^2-B^4+14AB+45B^2=0$
8	$A^8+BA^7-175A^6-127BA^5+(32B^2+12086)A^4$ $+4918BA^3+(-4056B^2-372519)A^2+(175B^3$ $-55404B)A+(-B^4+112995B^2+4251528)=0$
10	$-A^4+(B^7-7B^5-46B^3-497B)A^3+(40B^{10}$ $+270B^8-4460B^6-17980B^4-43065B^2+70325)A^2$ $+(380B^{13}+14109B^{11}-47814B^9-610996B^7+73551B^5$ $+4268337B^3+4866725B)A+(-B^{18}+378B^{16}-39735B^{14}$ $+1096626B^{12}-5671107B^{10}-13298598B^8+91426794B^6$ $+34490691B^4-279117675B^2-221445125)=0$
11	$B=A^5+11A^4+51A^3+121A^2+144A+66$
13	$A=B^7-13B^6+45B^5+52B^4-493B^3+351B^2+1215B-1404$
17	$A^4=B^3-17AB^2+34A^2B+34A^3-238B^2-442AB-389A^2$ $+1244B+1428A-1556$

Note that the modular equations of degree 5, 7 and 11 are equivalent to those arising from Ramanujan's alternative cubic theory as recounted in equations (7.22), (7.27) and (7.33) of [3]. In fact Ramanujan was the first to calculate 'signature three' functions in his lost notebook. Ramanujan's various claims on signature three invariants were established by Berndt, Chan, Kang, and Zhang [4]. Further values were established by Chan, Gee, and Tan [7], and by Chan, Liaw and Tan [8].

In the following sections we find that modular equations of numerous higher 'signatures' (what we have chosen to call levels), also exist. This prompts the question of whether a higher analogue of Ramanujan's own work involving the hypergeometric function, can be found. In fact Ramanujan investigated various of the generalized Weber functions but the only results we are aware of, relate them to theta functions. Many such beautiful identities are proved in the work of Evans [10].

5. Schläfli-type Modular Equations for Level Five Functions

Here the signs for the Fricke involution depend only on m modulo 5, however for $n=5$ we have $s(n)=6$, thus we need to look at m modulo 30.

For $m \equiv 3, 7, 13, 17, 23, 27 \pmod{30}$ we have $\left(\frac{n}{m}\right)=-1$ and we have to apply the solution for the problem this causes, as recounted in Section 2.5.

We define

$m \pmod{30}$	A	B
$1,7,13,19$	$(uv)^3+5^3/(uv)^3$	$(v/u)+(u/v)$
$2,8,14,26$	$(uv)+5/(uv)$	$(v/u)^3+(u/v)^3$
$3,9,21,27$	$(uv)^3+5^3/(uv)^3$	$(v/u)^3+(u/v)^3$
$4,16,22,28$	$(uv)^3+5^3/(uv)^3$	$(v/u)+(u/v)$
$6,12,18,24$	$(uv)^3+5^3/(uv)^3$	$(v/u)^3+(u/v)^3$
$11,17,23,29$	$(uv)+5/(uv)$	$(v/u)^3+(u/v)^3$

We obtain the following modular equations.

m	Level 5 Modular Equation
2	$A = B$
3	$A^2 - B^4 + 18AB + 85B^2 = 0$
4	$AB - B^6 + 13B^4 - 33B^2 - 4 = 0$
6	$-A^6 + (B^5 - 5B^3 - 49B)A^5 + (36B^6 - 150B^4 - 1155B^2 + 899)A^4$ $+(486B^7 - 2227B^5 - 11842B^3 + 38596B)A^3 + (2964B^8 - 28986B^6$ $-12267B^4 + 760530B^2 - 148528)A^2 + (7605B^9 - 257230B^7 + 375555B^5$ $+8153220B^3 - 4205920B)A + (-B^{12} + 5484B^{10} - 908490B^8$ $+603055B^6 + 37436835B^4 - 35206256B^2 + 8952064) = 0$
7	$A^2 - B^8 + 14AB^3 + 43B^6 - 70AB - 475B^4 + 1325B^2 = 0$
8	$A^8 + BA^7 - 59A^6 - 35BA^5 + (16B^2 + 1362)A^4 + 418BA^3 + (-380B^2$ $-14535)A^2 + (55B^3 - 1560B)A + (-B^4 + 2835B^2 + 60500) = 0$
9	$-B^6 + 460B^5 + (765A - 38578)B^4 + (243A^2 - 7920A + 344020)B^3$ $+(27A^3 - 891A^2 + 28620A - 1085828)B^2 + (A^4 - 54A^3 + 796A^2$ $-56520A + 1418560)B + (-A^4 + 36A^3 - 148A^2 + 31680A - 805376) = 0$
11	$B^2 - A^5 - 11A^2B - 11A^4 - 110AB - 30A^3 - 275B - 125A - 629 = 0$
13	$A^4 - B^{14} + 26A^3B^3 + 221A^2B^6 + 624AB^9$ $+274B^{12} - 78A^3B - 1066A^2B^4 - 4264AB^7 - 21267B^{10}$ $-1859A^2B^2 - 6760AB^5 + 516752B^8 + 11200A^2 + 62400AB^3$ $-5189595B^6 - 26000AB + 24476050B^4 - 54513625B^2 + 46962500 = 0$

6. Schläfli-type Modular Equations for Level Seven Functions

For $n = 7$ we must consider congruence classes of m modulo 28. We define

$m \pmod{28}$	A	B
$1, 9, 25$	$(uv)^2 + 7^2/(uv)^2$	$(v/u) + (u/v)$
$3, 19, 27$	$(uv) + 7/(uv)$	$(v/u)^2 + (u/v)^2$
$5, 13, 17$	$(uv)^2 + 7^2/(uv)^2$	$(v/u) - (u/v)$
$11, 15, 23$	$(uv) + 7/(uv)$	$(v/u)^2 - (u/v)^2$
m even	$(uv)^2 + 7^2/(uv)^2$	$(v/u)^2 + (u/v)^2$

We obtain the following modular equations.

m	Level 7 Modular Equation
2	$A - B^3 + 11B = 0$
3	$A = B - 3$
4	$A^3B - B^6 + 16A^2B^2 + 88AB^3 + 165B^4 + 8A^2 + 77AB + 203B^2 = 0$
5	$A = B^3 - 5B^2 + 3B - 5$
6	$B^{12} - 1524B^{10} - 2178AB^9 + (-1040A^2 + 22054)B^8 + (-228A^3$ $+13133A)B^7 + (-24A^4 + 4153A^2 + 15786)B^6 + (-A^5 + 857A^3$ $+34556A)B^5 + (100A^4 + 19468A^2 - 664375)B^4 + (5A^5 + 4445A^3$ $-267725A)B^3 + (518A^4 - 50475A^2 + 871875)B^2 + (31A^5 - 4650A^3$ $+174375A)B + (A^6 - 225A^4 + 16875A^2 - 421875) = 0$
9	$-B^6 + 17B^5 - 49B^4 + (9A - 124)B^3 + (-6A + 145)B^2$ $+(A^2 - 42A + 425)B + (A^2 - 30A + 229) = 0$
11	$B^6 - 66AB^5 + (1023A^2 - 38538)B^4 + (-2A^5 - 2130A^3 + 55764A)B^3$ $+(-55A^6 + 3861A^4 - 39204A^2 - 45927)B^2 + (66A^7 - 2250A^5 + 20538A^3$ $+91854A)B + (A^10 - 48A^8 + 758A^6 + 1440A^4 - 45927A^2) = 0$
13	$A^3 - B^7 + 13A^2B^2 + 52AB^4 + 39B^6 - 39AB^3 - 345B^5 + 13A^2 + 117AB^2$ $-65B^4 + 195AB + 1299B^3 - 121A - 1105B^2 + 2255B - 1573 = 0$

7. Schläfli-type Modular Equations for Level Eleven Functions

We define the following functions:

m	A	B
$\equiv 4 \pmod 6$	$(uv)^6 + 11^6/(uv)^6$	$(v/u)^2 + (u/v)^2$
$\equiv 2 \pmod 6$	$(uv)^2 + 11^2/(uv)^2$	$(v/u)^6 + (u/v)^6$
$\equiv 0 \pmod 6$	$(uv)^6 + 11^6/(uv)^6$	$(v/u)^6 + (u/v)^6$
$\equiv 9 \pmod{12}$	$(uv)^6 + 11^6/(uv)^6$	$(v/u)^3 + \left(\frac{11}{m}\right)(u/v)^3$
$\equiv 3 \pmod{12}$	$(uv)^3 + 11^3/(uv)^3$	$(v/u)^6 + \left(\frac{11}{m}\right)(u/v)^6$
$\equiv 1 \pmod{12}$	$(uv)^6 + 11^6/(uv)^6$	$(v/u) + \left(\frac{11}{m}\right)(u/v)$
$\equiv 5 \pmod{12}$	$(uv)^2 + 11^2/(uv)^2$	$(v/u)^3 + \left(\frac{11}{m}\right)(u/v)^3$
$\equiv 7 \pmod{12}$	$(uv)^3 + 11^3/(uv)^3$	$(v/u)^2 + \left(\frac{11}{m}\right)(u/v)^2$
$\equiv 11 \pmod{12}$	$(uv) + 11/(uv)$	$(v/u)^6 + \left(\frac{11}{m}\right)(u/v)^6$

We obtain the following modular equations.

m	Level 11 Modular Equation
2	$-448A^3 + 20416B^3 + 832A^2B + 11968AB^2 + A^5$ $-B^5 - 5A^4B + 10A^3B^2 - 10A^2B^3 + 5AB^4 = 0$
3	$A^{10} - 10BA^9 + (45B^2 - 10692)A^8 - (120B^3 - 117612B)A^7 + (210B^4$ $-3496041B^2 + 30233088)A^6 - (252B^5 - 30560652B^3 + 400588416B)A^5$ $+(210B^6 + 179527428B^4 + 18861667776B^2 - 8707129344)A^4 - (120B^7$ $+1091236212B^5 + 183316439520B^3 - 309103091712B)A^3 + (45B^8$ $-1976043924B^6 + 2518971763392B^4 - 5459370098688B^2)A^2 - (10B^9$ $+598446792B^7 + 13227440833728B^5 - 133802456629248B^3)A + (B^{10}$ $-27756432B^8 + 26159357011392B^6 - 614227025313792B^4) = 0$
5	$A^5 - B^5 - 5A^4B + 5AB^4 + 10A^3B^2 - 10A^2B^3 + 3220AB^3 - 1030A^2B^2$ $+70A^3B + 895B^4 - 30A^4 + 35175AB^2 + 1575A^2B - 73975B^3 - 275A^3$ $-9000AB - 358000B^2 + 17000A^2 - 170000B - 180000A + 550000 = 0$

8. Schläfli-type Modular Equations for Level Thirteen Functions

Taking into account the change of signs required when $\left(\frac{13}{m}\right) = -1$ and applying the usual solution to the q-series problem which arises, we actually find that in all cases the definitions of the functions A and B are the same.

m	A	B
All cases	$(uv) + 13/(uv)$	$(v/u) + (u/v)$

Of course for $m \equiv 5, 7, 11, 15, 19, 21 \pmod{26}$ we need to choose our modular equations so that all monomials have degrees with the same parity.

We have found the following modular equations.

m	Level 13 Modular Equation
2	$A = B^3 - 7B$
3	$A = B^2 - 3B - 5$
4	$BA^3 + (8B^2 + 4)A^2 + (28B^3 - 7B)A + (-B^6 + 45B^4 - 41B^2 - 100) = 0$
5	$A^4 - B^6 + 10A^3B + 45A^2B^2 + 100AB^3 + 106B^4$ $-32A^2 - 160AB - 329B^2 + 260 = 0$
6	$A^6 + (-B^5 + 5B^3 + 13B)A^5 + (-12B^6 + 50B^4 + 133B^2 - 79)A^4$ $+(-66B^7 + 263B^5 + 620B^3 - 607B)A^3 + (-196B^8 + 767B^6 + 1859B^4$ $-3603B^2 + 1854)A^2 + (-315B^9 + 1280B^7 + 2939B^5 - 9237B^3 + 7362B)A$ $+(B^{12} - 228B^{10} + 938B^8 + 2599B^6 - 12843B^4 + 18414B^2 - 8775) = 0$
7	$A^6 - B^8 + 14A^5B + 91A^4B^2 + 336A^3B^3 + 735A^2B^4$ $+882AB^5 + 463B^6 - 50A^4 - 448A^3B - 1736A^2B^2 - 3108AB^3$ $-2211B^4 + 625A^2 + 2450AB + 2725B^2 = 0$

9. Schläfli-type Modular Equations for Level Seventeen Functions

Here $s(n) = 3$, however the value j that must be used is twice this, since $n = 17$ is the exceptional value of Theorem 2.1.2.

We find that the functions A and B should be defined as follows.

m	A	B
$\equiv 2 \pmod 6$	$(uv) + 17/(uv)$	$(v/u)^3 + (u/v)^3$
$\equiv 0, 3, 5 \pmod 6$	$(uv)^3 + 17^3/(uv)^3$	$(v/u)^3 + (u/v)^3$
$\equiv 1, 4 \pmod 6$	$(uv)^3 + 17^3/(uv)^3$	$(v/u) + (u/v)$

Note that for $m \equiv 3, 5, 7, 11, 23, 27, 29, 31 \pmod{34}$ we must choose the monomials of our modular equations to have degrees of the same parity.

m	Level 17 Modular Equation
2	$A^2 - 2AB + B^2 + 4A - 20B - 32 = 0$
3	$B^{16} - 138202B^{14} - 14886AB^{13} + (-4A^2 + 3783484905)B^{12}-$ $1034090118AB^{11} + (26326263A^2 - 8314008632212)B^{10} + (-74736A^3$ $-637286076000A)B^9 + (6A^4 - 23029114566A^2 + 364651943394712)B^8$ $+(-403940250A^3 + 400406333328264A)B^7 + (-3816282A^4$ $+1978275107488A^2 - 994742608565088)B^6 + (-15822A^5 + 56551241664A^3$ $-92053329385248A)B^5 + (-4A^6 + 1039573617A^4 - 3595116947688A^2$ $+701428933357200)B^4 + (12639996A^5 - 74431249920A^3$ $+51577742606400A)B^3 + (99646A^6 - 885887280A^4$ $+1426156696800A^2)B^2 + (468A^7 - 5808672A^5 + 16344763200A^3)B$ $+(A^8 - 16200A^6 + 65610000A^4) = 0$
4	$B^{12} - 36B^{11} + 478B^{10} - 2800B^9 + 5851B^8 + (-2A + 8212)B^7$ $+(-24A - 50242)B^6 + (206A + 35280)B^5 + (-336A + 97041)B^4$ $+(-486A - 113956)B^3 + (A^2 + 1504A - 69688)B^2 + (-4A^2 - 396A$ $+76552)B + (4A^2 - 776A + 37440) = 0$

10. Polynomial Degrees of Modular Equations

As mentioned in the introduction, the minimal polynomial relationship between the functions $u(\tau)$ and $v(\tau) = u(m\tau)$ is often given by the Schläfli modular equation after we write A and B in terms of u and v and clear denominators. In some cases however, these equations are not the same.

In the following table we list the degrees d_{min} of the minimal polynomials relating u and v (the degree in u is always the same as that in v) and the degrees d_{sch} of the polynomials induced by the Schläfli modular equations which we have found, for the various levels and degrees investigated.

In describing the degrees, we will make use of the function

$$\psi(n) = n \prod_{p \mid n} \left(1 + \frac{1}{p}\right),$$

where the product is over all primes p dividing n.

Level	Degree	d_{sch}	d_{min}
2	9	$\psi(9)\cdot 3$	$\psi(9)\cdot 3$
	15	$\psi(15)\cdot 3$	$\psi(15)\cdot 3$
3	2	$\psi(2)\cdot 4$	$\psi(2)\cdot 4$
	4	$\psi(4)\cdot 4$	$\psi(4)\cdot 4$
	5	$\psi(5)$	$\psi(5)$
	7	$\psi(7)\cdot 2$	$\psi(7)$
	8	$\psi(8)\cdot 4$	$\psi(8)\cdot 4$
	10	$\psi(10)\cdot 4$	$\psi(10)\cdot 4$
	11	$\psi(11)$	$\psi(11)$
	13	$\psi(13)$	$\psi(13)$
	17	$\psi(17)$	$\psi(17)$
5	2	$\psi(2)\cdot 2$	$\psi(2)\cdot 2$
	3	$\psi(3)\cdot 6$	$\psi(3)\cdot 3$
	4	$\psi(4)\cdot 2$	$\psi(4)\cdot 2$
	6	$\psi(6)\cdot 6$	$\psi(6)\cdot 6$
	7	$\psi(7)\cdot 2$	$\psi(7)$
	8	$\psi(8)\cdot 2$	$\psi(8)\cdot 2$
	9	$\psi(9)\cdot 3$	$\psi(9)\cdot 3$
	11	$\psi(11)$	$\psi(11)$
	13	$\psi(13)\cdot 2$	$\psi(13)$

Level	Degree	d_{sch}	d_{min}
7	2	$\psi(2)\cdot 4$	$\psi(2)\cdot 4$
	3	$\psi(3)$	$\psi(3)$
	4	$\psi(4)\cdot 4$	$\psi(4)\cdot 4$
	5	$\psi(5)$	$\psi(5)$
	6	$\psi(6)\cdot 4$	$\psi(6)\cdot 4$
	9	$\psi(9)$	$\psi(9)$
	11	$\psi(11)\cdot 2$	$\psi(11)$
	13	$\psi(13)$	$\psi(13)$
11	2	$\psi(2)\cdot 20$	$\psi(2)\cdot 20$
	3	$\psi(3)\cdot 30$	$\psi(3)\cdot 15$
	5	$\psi(5)\cdot 5$	$\psi(5)\cdot 5$
13	2	$\psi(2)\cdot 2$	$\psi(2)\cdot 2$
	3	$\psi(3)$	$\psi(3)$
	4	$\psi(4)\cdot 2$	$\psi(4)\cdot 2$
	5	$\psi(5)\cdot 2$	$\psi(5)$
	6	$\psi(6)\cdot 2$	$\psi(6)\cdot 2$
	7	$\psi(7)\cdot 2$	$\psi(7)$
17	2	$\psi(2)\cdot 4$	$\psi(2)\cdot 4$
	3	$\psi(3)\cdot 24$	$\psi(3)\cdot 12$
	4	$\psi(4)\cdot 4$	$\psi(4)\cdot 4$

11. Evaluation of an Eta Quotient Using Modular Equations

In this final section we give a very simple example of an application of the modular equations we have developed. We explicitly evaluate a specific quotient of the Dedekind eta function. Evaluation of such quantities is important in obtaining explicit generators of ring class fields in explicit class field theory (see [12] for further details).

Our example will come from level three functions. In particular we will make use of the modular equation of degree five for this level.

We will specialise this modular equation by making the specific assignment $\tau = 1 - 1/\sqrt{-5}$. Then $5\tau = 5 - 5/\sqrt{-5} = 5 + \sqrt{-5}$.

Plugging this value of τ into the modular equation of degree five and level 3 we will end up with a polynomial relation between

$$\mathfrak{g}_0(\tau) = \mathfrak{g}_0(1 - 1/\sqrt{-5}) = \zeta_{12}^{-1}\,\mathfrak{g}_1(-1/\sqrt{-5}) = \zeta_{12}^{-1}\,\mathfrak{g}_2(\sqrt{-5})$$

and

$$\mathfrak{g}_0(5\tau) = \mathfrak{g}_0(5 + \sqrt{-5}) = \zeta_{12}^{-2}\,\mathfrak{g}_2(\sqrt{-5}).$$

Letting u be the first of these values and v the second, the appropriate modular equation becomes

$$(uv)^2 + 9/(uv)^2 - (v/u)^3 + (u/v)^3 + 5 = 0.$$

In other words

$$-\mathfrak{g}_2(\sqrt{-5})^4 - 9/\mathfrak{g}_2(\sqrt{-5})^4 + 2i + 5 = 0.$$

Finally if we let $x = \mathfrak{g}_2(\sqrt{-5})^4$ then rearranging and squaring the previous expression yields the following irreducible polynomial equation:

$$x^4 - 10x^3 + 47x^2 - 90x + 81 = 0.$$

Noting that

$$x = \mathfrak{g}_2(\sqrt{-5})^4 = \frac{\eta((\sqrt{-5}+2)/3)^4}{\eta(\sqrt{-5})^4},$$

we see that we have completed an evaluation of a non-trivial eta quotient.

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